Problem Set 3

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 3 covers materials from §2.4 – §3.2.

1. Consider a unit mass located at a distance r from the center of the Earth. The size of the gravitational force $F(r)$ exerted by the Earth on the mass is
$$F(r)=\begin{cases}\dfrac{GMr}{R^3} & \text{if } 0\le r<R\\ \dfrac{GM}{r^2} & \text{if } r\ge R\end{cases},$$
where M is the mass of the Earth, R is the radius of the Earth, and G is the gravitational constant. Is this function F continuous or not?

2. Let a be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by
$$f(x)=\begin{cases}e^x+\dfrac{\sin 2x}{x} & \text{if } x<0\\ 3\sin x+ae^x & \text{if } x\ge 0\end{cases}.$$
Find the value(s) of a such that f is continuous on $\mathbb{R}$.

3. Evaluate each of the following limits, if it exists:
   (a) $\displaystyle\lim_{x\to\frac{\pi}{2}}\sin(\tan(\cos x))$
   (b) $\displaystyle\lim_{x\to1}\arcsin\frac{1-\sqrt{x}}{1-x}$
   (c) $\displaystyle\lim_{x\to+\infty}2^{\frac{\sin x}{x}}$
   (d) $\displaystyle\lim_{x\to0}\ln\left(1+x^2\cos\frac{1}{x}\right)$

4. (a) Evaluate the limit
$$\lim_{x\to+\infty}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}.$$
   (b) Using the result from (a) or otherwise, evaluate the limit
$$\lim_{x\to+\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right).$$

5. Let a be a real number and f be a function. Show that
$$\lim_{x\to a}f(x)=0 \qquad \text{if and only if} \qquad \lim_{x\to a}|f(x)|=0$$
using the fact that the absolute value function is continuous.

6. Let f be a polynomial given by
$$f(x)=x^n-a_{n-1}x^{n-1}-a_{n-2}x^{n-2}-\cdots-a_1x-a_0.$$
If $a_0+a_1+\cdots+a_{n-1}<1$ and $a_0>0$, show that f has a root in $(0,1)$.

7. Let $f: [0, 2] \to \mathbb{R}$ be a continuous function with
$$f(0) = f(2).$$
Show that there exists $c \in [0, 1]$ such that
$$f(c) = f(c + 1).$$

8. For each of the following equations, show that there is a solution in the given interval.
   (a) $x^3 - 4x - 1 = 0$, interval: $(-1, 0)$
   (b) $\ln x = x - \sqrt{x}$, interval: $(2, 3)$
   (c) $\cos\sqrt{x} = e^x - 2$, interval: $(0, 1)$

9. Let $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be the function defined by
$$f(x) = \frac{x + 1}{x - 1}.$$
Find the derivative of f **from definition**.

10. Let $f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ be the function defined by
$$f(x) = \csc 2x\,.$$
Find the derivative of f **from definition**.

11. Prove that
$$\frac{d}{dx}\ln(1 + x) = \frac{1}{1 + x}$$
from the definition of derivative. What is the domain of this derivative?

12. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}.$$
   (a) Find $f'(x)$ for each $x \neq 0$ **from the definition of derivative**.
   (b) Compute $\lim_{x \to 0} f'(x)$.
   (c) Determine whether f is differentiable at 0 or not, and evaluate $f'(0)$ if it is.

13. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. **Without using the chain rule**, prove that
   (a) If f is even, then f' is odd.
   (b) If f is odd, then f' is even.
   (c) If f is periodic, then f' is also periodic.

14. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} \dfrac{2x}{1 + e^{1/x}} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b(\cos x + \sin x) + c & \text{if } x > 0 \end{cases}.$$
If f is differentiable at 0, find the values of a, b and c.

15. On a piece of graph paper, sketch the graph of a function f which satisfies all the following conditions:
    (i) The domain of f is $[-3,3]$, and the range of f is $[-2,2]$.
    (ii) f is even.
    (iii) f is continuous on $[-3,3]$.
    (iv) f is differentiable at 0.
    (v) $f'(x) > 0$ for every $x \in (0,2)$.
    (vi) $\lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = +\infty$.
    (vii) $f'(x) < 0$ for every $x > 2$.
    (viii) $\lim\limits_{x\to 3^-} \dfrac{f(x)}{x-3} = -1$.

16. Let f be the function

$$f(x) = \frac{5x}{1+x^2}.$$

    (a) Find an equation of the tangent line to the graph of f at the point $\big(2, f(2)\big)$.
    (b) Find an equation of the normal line to the graph of f at the point $\big(3, f(3)\big)$.

17. Consider a quadratic polynomial

$$f(x) = ax^2 + bx + c.$$

In Example 1.17, we recalled that the signs of a, b and c can be read from the graph of f. In particular,

⊙ $b = -2ah$ where h is the x-coordinate of the vertex of the parabola.

Now show that we have an even easier graphical interpretation of b as follows:

⊙ b is the slope of the tangent line to the parabola at the point where it intersects the y-axis.

18. Find the derivative of each of the following functions.
    (a) $f(x) = \dfrac{x^{1013}+1}{1013x+1}$
    (b) $f(x) = \dfrac{x}{x+\dfrac{2}{x}}$
    (c) $f(x) = \dfrac{x}{\sin x} + \dfrac{\sin a}{a}$, where a is a non-zero real constant.
    (d) $f(x) = \dfrac{x}{\sec x + \tan x}$